

# Inverse transfer of self-similar decaying turbulent non-helical magnetic field

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## Abstract

We show that decaying turbulent non-helical magnetic fields satisfy a self-similarity relation according to which the relevant scales increase as time passes (inverse cascade or inverse transfer). We compute analytically quantities which have previously been determined by numerical calculations, for example the average energy and the integral scale which are proportional to  $1/t$  and  $\sqrt{t}$ , respectively, where  $t$  is time. We also briefly discuss self-similarity for the helical case.

The important subject of freely decaying non-helical magnetic fields have recently caught much interest [1],[2]. These fields are of interest for cosmology (for a recent review, see [3]) and for the interpretation of new measurements of optical polarization in relation to gamma ray burst afterglows (for a recent discussion, see [4]). The interesting feature of the investigations [1] and [2] is that there is an inverse transfer (also called an inverse cascade<sup>1</sup>) from small to large scales of the magnetic field, in spite of the fact that there is no magnetic helicity.

In this note we shall show that some of the results obtained numerically in refs. [1] and [2] can be understood analytically quite rigorously in magnetohydrodynamics (MHD). We shall find that the kinetic and magnetic energy spectra are self-similar, as found numerically for the magnetic field energy by Zrake<sup>2</sup> [2]. We start by giving the main result,

$$\mathcal{E}_{v,B}(k, t) = \sqrt{\frac{t_0}{t}} \mathcal{E}_{v,B} \left( k \sqrt{\frac{t}{t_0}}, t_0 \right), \quad (1)$$

where  $\mathcal{E}_{v,B}$  are the kinetic and magnetic energy densities normalized as

$$\int_0^\infty dk \mathcal{E}_v = \frac{1}{2} \langle \mathbf{v}(\mathbf{x}, t)^2 \rangle, \quad \int_0^\infty dk \mathcal{E}_B = \frac{1}{2} \langle \mathbf{B}(\mathbf{x}, t)^2 \rangle \quad (2)$$

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<sup>1</sup>Some people consider this terminology to be politically incorrect and it should be reserved for the helical case only.

<sup>2</sup>His calculation is done in relativistic MHD, whereas our result refers to non-relativistic MHD

for isotropic turbulence. The scaling exhibited in Eq.(1) is expanding in the sense that the effective  $k$  is shifted towards  $k = 0$ , i.e. towards larger distances.

Eq. (1) is valid when the unforced MHD equations are satisfied. If the initial conditions are established by some initial forcing (“stirring”) then the self-similarity (1) only sets in after the switching off of the external forces. The actual form of the initial conditions is of no relevance for the self-similarity. Our result is valid for the non-relativistic version of the MHD equations. This may explain a slight discrepancy between Eq. (1) and the result in ref. [2]. The prefactor  $\sqrt{t_0/t}$  in (1) is slightly different from the one obtained by Zrake, and the power of  $t/t_0$  associated with  $k$  is also slightly different.

From (1) we immediately obtain the time dependence of the integrated energy density (we assume that  $\mathcal{E}$  is not a constant)

$$\mathcal{E}_{v,B}(t) = \int_0^\infty dk \mathcal{E}_{v,B}(k, t) = \frac{t_0}{t} \int_0^\infty dx \mathcal{E}_{v,B}(x, t_0) \propto \frac{1}{t}. \quad (3)$$

This is in accordance with the numerical results obtained by Kahnishvili, Brandenburg, and Tevzadze [1]. As we shall see from the derivation of (1) this result is rigorous and is thus also a check of the numerical calculations. Similarly we can obtain the kinetic or magnetic integral scale

$$\langle \xi \rangle = \kappa^{-1}(t) = \frac{\int_0^\infty \frac{dk}{k} \mathcal{E}(k, t)}{\mathcal{E}(t)} = \sqrt{\frac{t_0}{t}} \frac{\int_0^\infty \frac{dx}{x} \mathcal{E}(x, t_0)}{\mathcal{E}(t)} \propto \sqrt{t}, \quad (4)$$

again in accordance with ref. [1]. Higher moments can be computed,

$$\langle \xi^2 \rangle \propto t, \quad \langle \xi^3 \rangle \propto t^{3/2}, \quad (5)$$

etc. For the higher moments there may be convergence problems at  $x = 0$ , so it may be more productive to consider moments

$$\langle k^n \rangle \propto t^{-n/2}. \quad (6)$$

The derivation of the main equation (1) follows to some extent an earlier paper [5] (see also [6]), except that we do not attempt to interpret the results in terms of initial values. The latter required some assumptions on inertial range behaviour. A critical discussion of problems in this connection is given in refs. [1] and [7].

To get (1) we start from the observation that the unforced MHD equations

$$\partial_t \mathbf{v} + (\mathbf{v} \nabla) \mathbf{v} = -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{v}, \quad \partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad (7)$$

are invariant under the scalings

$$\mathbf{x} \rightarrow l\mathbf{x}, \quad t \rightarrow l^2 t, \quad \mathbf{v} \rightarrow \mathbf{v}/l, \quad \mathbf{B} \rightarrow \mathbf{B}/l, \quad \nu \rightarrow \nu, \quad \eta \rightarrow \eta, \quad p \rightarrow p/l^2. \quad (8)$$

The crucial point is that the kinetic and Ohmic diffusions  $\nu$  and  $\eta$ , respectively, are invariant under this scaling. Let us consider the energy density

$$\mathcal{E}_v(k, t) = \frac{2\pi k^2}{(2\pi)^3} \int d^3 y e^{i\mathbf{k}\mathbf{y}} \langle \mathbf{v}(\mathbf{x}, t) \mathbf{v}(\mathbf{x} + \mathbf{y}, t) \rangle, \quad (9)$$

with a similar expression for the magnetic energy density with  $\mathbf{v}$  replaced by  $\mathbf{B}$  on the right hand side. It is easily seen that integration of both sides of Eq. (9) gives Eq. (2). We now replace the variables according to the scalings (8) to obtain ( $\mathbf{y} = l\mathbf{y}'$ )

$$\mathcal{E}_v(k/l, l^2t) = l \frac{2\pi k^2}{(2\pi)^3} \int d^3y' e^{i\mathbf{k}\mathbf{y}'} \langle \mathbf{v}(l\mathbf{x}', l^2t) \mathbf{v}(l(\mathbf{x}' + \mathbf{y}'), l^2t) \rangle = l^{-1} \mathcal{E}_v(k, t), \quad (10)$$

with a similar expression for  $\mathcal{E}_B$ . Thus

$$\mathcal{E}_{v,B}(k, t) = l \mathcal{E}_{v,B}(k/l, l^2t) \quad (11)$$

This is a functional relation for all values of the parameter  $l$ . We want now to compare the situation for two different times  $t_0$  and  $t$  by taking

$$l = \sqrt{t_0/t}, \quad (12)$$

leading immediately to Eq. (1). Here all considerations are valid for the full interval of  $k$ .

In an earlier paper [5] we solved Eq. (11) by

$$\mathcal{E}_{v,B}(k, t) = k\psi(k^2t), \quad (13)$$

which can be brought in the form of Eq. (1) by writing

$$k^2t = (k\sqrt{t/t_0})^2t_0. \quad (14)$$

The former approach [5],[6] also involves the use of the invariance of the MHD equations under the scalings

$$\mathbf{x} \rightarrow l\mathbf{x}, \quad t \rightarrow l^{1-h}t, \quad \mathbf{v} \rightarrow l^h\mathbf{v}, \quad \mathbf{B} \rightarrow l^h\mathbf{B}, \quad \nu \rightarrow l^{1+h}\nu, \quad \eta \rightarrow l^{1+h}\eta. \quad (15)$$

This scaling is physically somewhat unpleasant to apply for  $h \neq -1$  since it involves changing diffusions, i.e. moving between different universes. For further critical discussions of the approach in [5] and [6] we refer to refs.[1] and [7]. In particular the identification of the time  $t = 0$  as the initial time is doubtful because this scaling is not valid when the process of turbulence is initiated by some forces. Also  $t = 0$  may be a singular time [7].

Similar critical remarks and reinterpretations of ref.[5] were made by Campanelli [8]. He derived the result that for constant dissipation parameters the energy decays as  $1/t$  and the integral scale goes like  $\sqrt{t}$ . He also found cosmological uses of the case where the diffusions change as in the scaling Eq. (15).

The physical picture which emerges from the self-similarity (1) is that as time passes the physical scales increase. Any structure of size  $L$  at time  $t_0$  will expand into  $\sqrt{t/t_0}L$  at time  $t$ . This is similar to what happens in the expanding universe, with the factor  $\sqrt{t/t_0}$  being analogous to the scale factor in cosmology. The self-similar expansion is basically inherent in the dynamical properties of the MHD equations even in flat space. As discussed already in refs. [5] and [8] this can be extended quite simply to the expanding universe. In general the inverse transfer makes the decay of the magnetic field slower than dictated by the expansion

of the universe itself. This may be a more general phenomenon applicable to other vector fields, thereby making the conventional (text book) wisdom on the rapid decrease of vector fields relative to scalars somewhat insecure. Of course, a detailed analysis is needed for other vector fields in each case in order to see if they have inverse transfer.

Although we have considered the non-helical case it may be possible to generalize to the case where we have helicity, provided this kind of turbulence is really statistically isotropic, which may be somewhat doubtful. Assuming self-similarity we have in analogy with the construction in Eq.(9)

$$\mathcal{H}(k, t) = \frac{4\pi k^2}{(2\pi)^3} \int d^3y e^{i\mathbf{k}\cdot\mathbf{y}} \langle \mathbf{A}(\mathbf{x}, t) \mathbf{B}(\mathbf{x} + \mathbf{y}, t) \rangle. \quad (16)$$

Scaling now gives

$$\mathcal{H}(k, t) = \mathcal{H}(k\sqrt{t/t_0}, t_0) \quad (17)$$

In ideal MHD with  $\eta = 0$  the helicity is constant. However, in the presence of diffusion helicity decays. With

$$H(t) = \int_0^\infty dk \mathcal{H}(k, t) = \langle \mathbf{A}(\mathbf{x}, t) \mathbf{B}(\mathbf{x}, t) \rangle \quad (18)$$

we have

$$H(t) = \int_0^\infty dk \mathcal{H}(k, t) \propto 1/\sqrt{t}. \quad (19)$$

Thus helicity decreases slower with time than the energy  $\propto 1/t$ . As observed in ref. [1] there will always be some helicity, due to fluctuations. These should then follow the self-similarity (17).

As mentioned before self-similarity may not be right for the helical case, especially if one considers long time intervals where  $H$  is constant. As an example there are the results in ref. [9] according to which the energy decays as  $1/\sqrt{t}$  for the helicity  $H$  fixed. Such a behavior is clearly not covered by our results. A discussion of the problem of helicity versus scaling has been given by Campanelli [8].

The conclusion from this note is that the inverse transfer reported in the literature can in fact be understood as a result of the well known scaling relations (8), and the decay of the energy and the behavior of the integral scale as  $1/t$  and  $\sqrt{t}$ , respectively, are simple consequences of the self-similarity (1) resulting from these scaling relations. The inverse transfer is therefore a generic feature of freely decaying (magneto-)hydrodynamics.

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